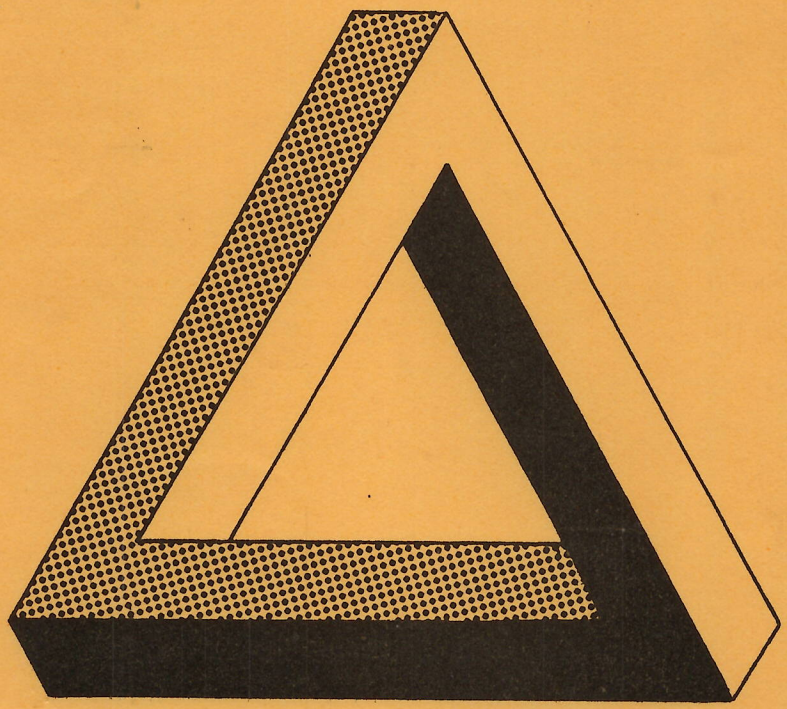
Hank De Witt

TRIGON



SCHOOL MATHEMATICS JOURNAL
OF
THE MATHEMATICAL ASSOCIATION OF
SOUTH AUSTRALIA

VOLUME 15, NO.1 MARCH, 1977

TRIGON

SCHOOL MATHEMATICS JOURNAL

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EDITORIAL COMMITTEE

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A number of back issues are still available:

School Mathematics Journal (up to 1974) - 15 cents per copy.

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READERS' CONTRIBUTIONS

The editorial committee is always delighted to hear from the readers. Letters, short articles, mathematical snippets and cartoons should be sent to

The Editor, TRIGON,
Department of Pure Mathematics,
University of Adelaide,
Adelaide, 5001.

All material which is suitable for publication will be acknowledged in the Journal.

PRIZES

Book prizes will be awarded to students who submit the most correct solutions in each section to problems set in this Magazine.

LETTER TO THE EDITOR

I was pleased to see some graph theory amongst the last set of senior problems (Vol.14, No.3). It was embedded in problem 036: represent each football team by a point and join two points by a line if and only if the respective football teams have played their match. Such a configuration of points and lines is called a graph. The question asked then becomes: show that for any graph

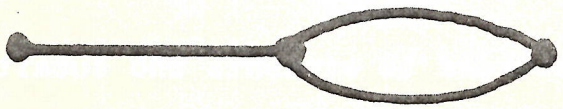
(1) at least two points have the same number of lines emanating from them (i.e., have the same degree)

(2) the number of points with odd degree is even.

[See solution to 036 in this issue, Editor.] The first result must have been found many times, but the earliest explicit record I know of is in the paper "No graph is perfect" by Behzad and Chartrand (American Mathematical Monthly, Vol.74, pp.962-963) published as late as 1967. (The proof is that with n points the only possible degrees are the n numbers $0,1,2,\ldots,n-1$ and if each point has a unique degree then, in particular, one will have degree 0 and another will have degree $n-1$. Thus one point is not joined to any other point, while another point is joined to all other points; a contradiction.) [Mark Lawrence gave essentially this proof for 036; a copy of the above paper can be obtained on request from the Editor of Trigon - Editor.]

Behzad and Chartrand go on to ask what graphs have just two points with the same degree. They prove by the Principle of Mathematical Induction that for any given number $n(\geq 2)$ there are just two such graphs with n points and one is the complement of the others (i.e., one has just those lines that the other has not). Moreover, their proof is constructive, i.e. it shows how to construct such graphs.

We should note that result (1) does not apply to multi-graphs, i.e. graphs in which more than one line between points is allowed. For example:



Hence the restriction in the football team that "every team is to play each team only once".

This restriction is not required for result (2) however. Also in contrast, this result is much older than the first and in fact belongs to the genesis of graph theory! It is an immediate corollary to the theorem that appears in the first paper on graphs, which was written by the famous mathematician Leonhard Euler in 1736. (The paper deals with the well-known Koenigsberg Bridge Problem. A translation appears in Mathematics in the Modern World, M. Kline ed., Freeman 1968.) The theorem is that the sum of the degrees of all points in a multigraph is even (simply because this is twice the number of lines in the multigraph); hence a direct contradiction if the number of points with odd degree is odd.

Of course, the original football problem can be solved quite adequately without a hint of graph theory. But in using it we attempt to carry out one of the roles of mathematics: to build an abstract model of a concrete situation not only to further our knowledge of that situation but also to extend it to other situations - in the case of graph theory: electrical circuits, transportation and communication networks, social relationships, molecules, etc....

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Mathematics Department,
Gordon Institute of Technology,
Geelong, Victoria.

BI-CENTENARY ESSAY COMPETITION

April 30, 1977, marks the two-hundredth anniversary of the birthday of KARL FRIEDRICH GAUSS. Gauss was one of the greatest mathematicians of all time and to commemorate this occasion, TRIGON is holding an essay competition. The essay competition is open to all students. Your essay, which should deal with some aspects of the life and work of Gauss, should be of approximately 2000 words. Book prizes will be awarded to the best original essay from each year, and the best essay will be published in TRIGON, Vol. 15, No. 3.

You should send your essay together with your year and the name of your school to the Editor of TRIGON, Department of Pure Mathematics, University of Adelaide, North Terrace, Adelaide, by August 12th, 1977.

GENERATING FUNCTIONS

by M. Sved

Part 1

The parents' association of the Euclid Park Primary School is a very active body. All fathers and mothers of the 200 children always attend every meeting and by the nature of things decision making is very difficult. The parents decide to form a working committee of 6 members:

Mrs. Appleby:- I propose we appoint 6 mothers.

Mr. Cherry:- This is sex-discrimination!

Mrs. Pearson:- Why should we not have a mixed committee: 3 fathers and 3 mothers?

Mr. Plumrose:- Too inflexible! Why not elect 6 married couples on the committee, so that any one of them, husband or wife, could turn up at the meetings at their convenience?

Mrs. Peach:- Surely there are couples amongst us where both husband and wife are such keen workers and contribute so much that it would be a good idea to appoint both.

Mr. Berry:- I wish to move that a committee of 6 be appointed. Married couples should be given the membership of the committee with anyone to attend a meeting at a given time. However, we should not rule out the possibility of giving double membership to some couples, with both husband and wife attending all meetings. The total number of committee members should be confined to 6 in any case.

Chair person:- Any one to second the motion?

The motion was passed. In fact the meeting is still in session, trying to appoint the committee. Of course, the number of possibilities is somewhat large.

Let us formulate the problem. There are n couples, that is $2n$ persons in the association. A committee of r members is to be formed; membership being awarded to couples. A single membership entitles one representative of the couple to attend any one meeting, while a double membership means that both husband and wife are to attend.

Denote by E_r^n (E for Euclid Park) the number of possible committees. In our case $n = 200$, $r = 6$. One natural way to approach the problem of evaluating E_r^n is by the method of recursion.

When $n = 1$, the problem is easy. We could have a subcommittee of 1 or 2 members only. Hence

$$E_1^1 = E_2^1 = 1 \quad \text{and} \quad E_r^1 = 0 \quad (\text{for } r \geqslant 3)$$

we define $E_0^n = 1$ for all values of n. (It is reasonable to say that there is 1 way of choosing a committee without members.) Furthermore E_r^n is different from zero for $0 \leqslant r \leqslant 2n$; otherwise $E_r^n = 0$. We claim that:

$$E_r^{n+1} = E_{r-2}^n + E_{r-1}^n + E_r^n \qquad \ldots(R)$$

Indeed, assume that the solution to the problem is known for the case of n couples. The parents of a new student appear on the scene. We want to find the number of committees of r members when there are $n + 1$ couples. There are three possibilities to be considered:

(a) Mr. and Mrs. Newberry are both appointed. (double membership)

(b) The Newberry couple is given a single membership.

(c) The Newberrys are ignored.

In case (a) there are $r - 2$ members to be found out of the original n couples.

In case (b) there are $r - 1$ members to be found out of the original n couples.

In case (c) there are r members to be found out of the original n couples.

This explains the recursion relation (R).

We may now evaluate E_r^n for any values of n and r, beginning with:

$$E_0^1 = E_1^1 = E_2^1 = 1$$

and using (R) for obtaining successively the other values. Since in our case $n = 200$, this is a somewhat cumbersome procedure.

Define a function, connected with the values E_0^n, E_1^n, ... in the following way:

$$G^n(x) = E_0^n + E_1^n x + E_2^n x^2 + \ldots + E_{2n}^n x^{2n} \qquad \ldots(1)$$

Note that for $r > 2n$, $E_r^n = 0$. Consider now the product

$$G^n(x).(1 + x + x^2)$$

$$= G^n(x) + G^n(x).x + G^n(x)\, x^2$$

$$= E_0^n + E_1^n x + E_2^n x^2 + \ldots + E_{2n}^n x^{2n}$$

$$+ E_0^n x + E_1^n x^2 + \ldots + E_{2n-1}^n x^{2n} + E_{2n}^n x^{2n+1}$$

$$+ E_0^n x^2 + \ldots + E_{2n-2}^n x^{2n} + E_{2n-1}^n x^{2n+1} + E_{2n}^n x^{2n+2}$$

$$= E_0^n + (E_0^n + E_1^n) x + (E_0^n + E_1^n + E_2^n) x^2 + \ldots$$

$$\ldots + (E_{r-2}^n + E_{r-1}^n + E_r^n) x^r + \ldots + E_{2n}^n x^{2n+2}$$

If we look at (R) and also check the first and last two terms, we find that:

$$G^n(x) = E_0^{n+1} + E_1^{n+1} x + E_2^{n+1} x^2 + \ldots + E_r^{n+1} x^r + \ldots + E_{2n+2}^{n+1} x^{2n+2}$$

This means that by defining $G^{n+1}(x)$ by replacing n by $n + 1$ in equation (1), we obtain:

$$G^{n+1}(x) = G^n(x).(1 + x + x^2)$$

Since: $G^1(x) = E_0^1 + E_1^1 x + E_2^1 x^2 = (1 + x + x^2)$,

it follows that:

$$G^2(x) = G^1(x).(1 + x + x^2) = (1 + x + x^2)^2$$

and similarly:

$$G^n(x) = (1 + x + x^2)^n$$

Hence our problem is solved if we can find the coefficient of x^r in $(1 + x + x^2)^n$.

In the particular case of the Euclid Park Primary School Parents' Association we look for E_6^{200} ; that is the coefficient of x^6 in

$$G^{200}(x) = (1 + x + x^2)^{200} = [(1 + x) + x^2]^{200}$$
$$= (1 + x)^{200} + \binom{200}{1}(1 + x)^{199} x^2$$
$$+ \binom{200}{2}(1 + x)^{198} x^4$$
$$+ \binom{200}{3}(1 + x)^{197} x^6 + \ldots$$

It can be seen that the coefficient of x^6 is

$$\binom{200}{6} + \binom{200}{1}\binom{199}{4} + \binom{200}{2}\binom{198}{2} + \binom{200}{3}$$
$$= 95,476,299,600.$$

Thus by the use of the functions $G^n(x)$ we achieved a considerable short cut in evaluating the solution. The parents' meeting at the Euclid Park Primary School was certainly a function which generated a not quite trivial combinatorial problem. However, the term "generating function" is used in a different sense in combinatorics: the science of enumerating arrangements. These arrangements depend on variables which are natural numbers such as n or r in the problem discussed. So we are interested in sequences of the type:

$$u_0, u_1, u_2, \ldots, u_n, \ldots \qquad \ldots(*)$$

With this sequence we associate the function:

$$G(x) = u_0 + u_1 x + u_2 x^2 + \ldots + u_n x^n + \ldots$$

Where the coefficient of the nth power of x is the nth term of the sequence. (*)

[We have used the Binomial theorem here:

$$(a + b)^n = a^n + \binom{n}{1} a^{n-1} b + \binom{n}{2} a^{n-2} b^2 + \ldots + \binom{n}{r} a^{n-r} b^r + \ldots + b^n$$

This theorem is discussed in the Year 11 text books.]

We call $b(x)$ the generating function of the sequence $\{u_n\}$.

A well known example for a class of generating functions is:

$$B^n(x) = (1 + x)^n = \binom{n}{0} + \binom{n}{1} x + \ldots + \binom{n}{r} x^r + \ldots + \binom{n}{n} x^n ,$$

where the sequence is that of the binomial coefficients

$$\binom{n}{r}$$

and where n is fixed and r ranges from 0 to n.

In the above examples the generating functions were simple polynomials, since the sequences considered were finite. However, in most enumerative problems, the sequences considered are infinite. Take for example the simple problem of distributing + and - signs in n positions. The number of possibilities is:

$$u_n = 2^n$$

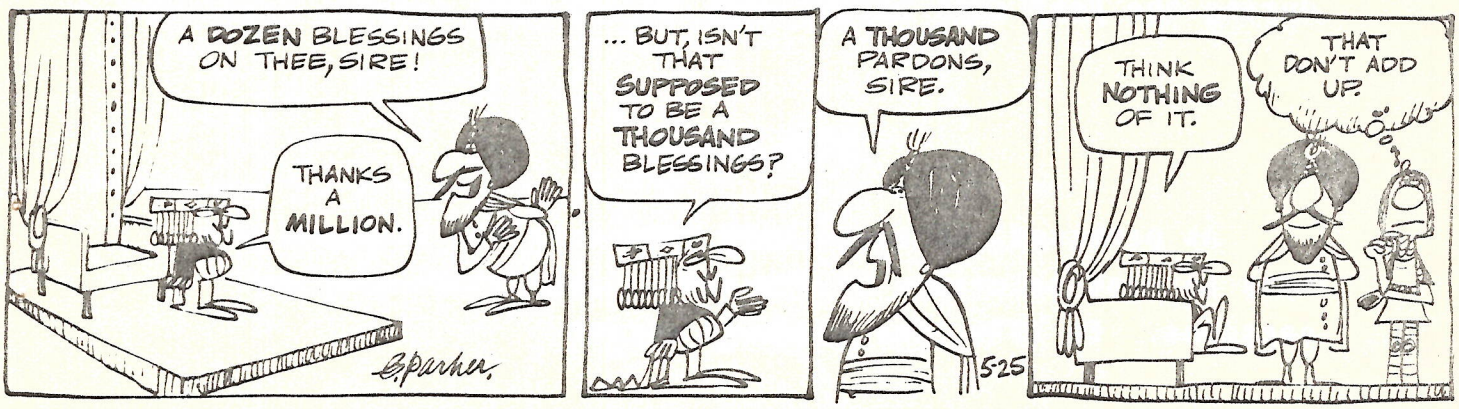
The corresponding generating function is:

$$G(x) = 1 + 2x + 2^2x^2 + \ldots + 2^nx^n + \ldots$$

Many of you will recognize this as an infinite geometric series. For a certain set of values of x (namely, $-\frac{1}{2} < x < \frac{1}{2}$) this series is convergent and

$$G(x) = \frac{1}{1 - 2x} .$$

We will discuss this situation with examples in part 2 of this article which will appear in the next issue.


A DOZEN BLESSINGS ON THEE, SIRE!
THANKS A MILLION.
... BUT, ISN'T THAT SUPPOSED TO BE A THOUSAND BLESSINGS?
A THOUSAND PARDONS, SIRE.
THINK NOTHING OF IT.
THAT DON'T ADD UP.
5-25

PROBLEMS

Solutions to Junior Problems will be accepted only from students in third or lower secondary years. Solutions of Problems in the Open Section will be accepted from students in any grade from Matriculation downwards. Some, but not all problems in the Open Section require more advanced knowledge than those in the Junior Section.

All solutions sent in must be the result of the solvers' unaided efforts.

Where a solution is obtained by the co-operation of two participants, the solution should be jointly signed by both authors. Our published solutions sometimes have to be abbreviated, but solutions sent in should be given fully.

Problems in this issue(Vol.15, No.1). We hope to publish these solutions and the names of those who sent in solutions in Vol.15, No.2. Solutions should be in the hands of the Editor by the end of April, 1977.

IMPORTANT. The solution of each problem must be written on a separate sheet, but the sheets may be sent in one envelope. Each sheet should be signed by the author(s), giving full name (in block letters), name of school and grade. You may send in solutions to one problem only or to any number of problems.

Solutions should be sent to The Editor, TRIGON, Department of Pure Mathematics, University of Adelaide, Adelaide, 5001.

Book prizes will be awarded to those students who have during the year sent in the most or the best correct solutions.

BOOK PRIZES

As promised, we award book prizes to the students who sent in most solutions to the Junior and Senior problem sections. In 1976, the students were:

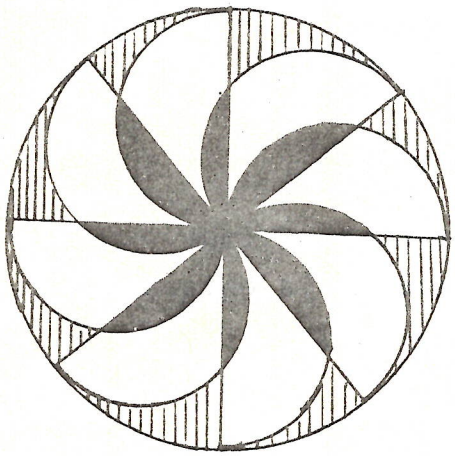
Senior: Mark Lawrence (Year 11) Westminster School

Junior: Kirsten Steele (Year 10) Walford C.E.G.G.S.

JUNIOR PROBLEMS

43. You would not add two fractions $\frac{a}{b}$ and $\frac{c}{d}$ to obtain $\frac{a+c}{b+d}$. Can you give at least two examples when the sum is correct? Is it possible that the sum is correct when each of a,b,c and d is positive? What can happen if we specify only that $\frac{a}{b}$ and $\frac{c}{d}$ are both positive?

44.



The circumference of a circle is divided up into n equal arcs and the points of division are joined to the centre. A semi-circle is drawn on each of the radii shown to obtain the pattern shown. Is it possible to choose n so that the dark shaded area (of the flower) is equal to the lightly shaded area of the edge?

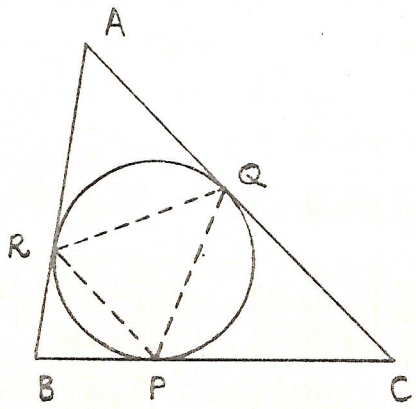
45. No family has more than 4 children in the Euclid Park Boys High School. One day when all the boys are at school, they answer a questionnaire. One of the questions is: How many brothers have you got at the school? In the subsequent statistics put out by the school, this statement appears:

216 boys have no brother or two brothers at the school

195 boys have one or three brothers at the school.

Can this statement be correct?

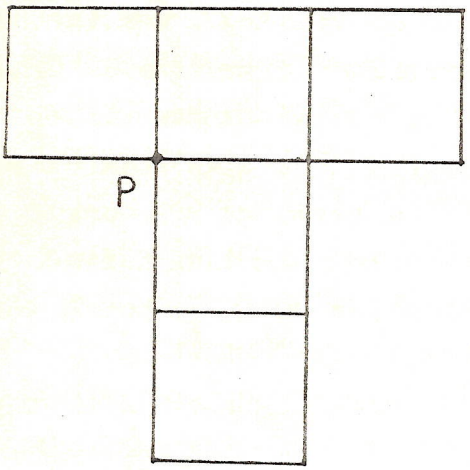
46.



The circle inscribed in a triangle ABC touches the sides at P,Q and R. Is it possible that the triangle PQR is similar to triangle ABC. (In other words can triangles ABC and PQR have the same angles.)

47. Find all integers (positive or negative) x and y such that

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{2}$$

48. 

The figure shown has an area of 5 square units. How many straight lines can be drawn through P such that they divide the figure into two sections of equal area? Prove your result.

49. Show that by adding one to the product of four consecutive integers, a perfect square is obtained. [e.g. $2 \times 3 \times 4 \times 5 + 1 = 121 = 11^2$]

SENIOR PROBLEMS

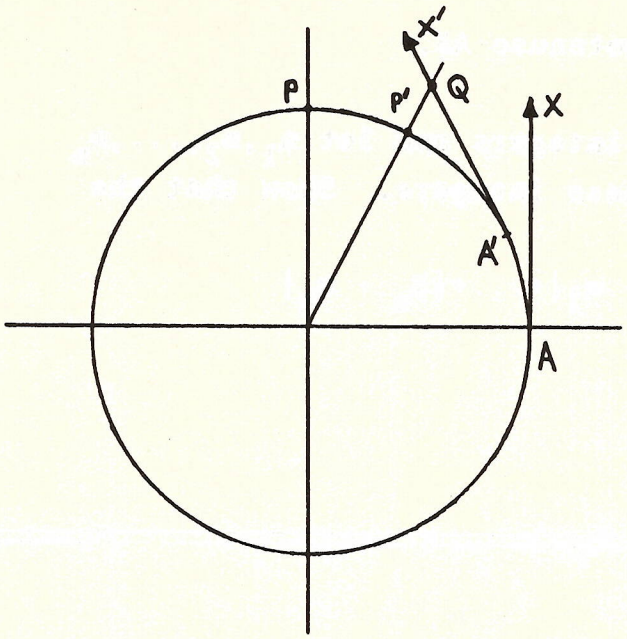
043. Refer to Problem 44 (Junior Section). What happens to the shaded border as $n \to \infty$ (n tends to infinity)? Prove your statement.

044. Find the number which has 8 divisors, such that the product of the divisors is 331776.

045. Show that $2^n - 1$ cannot be a perfect power. (That is, of the form a^k where a,k are positive integers and $k \geq 2$.)

046. OA,OP are two radii of a circle perpendicular to each other at $t = 0$. (t = time) A tangent-line AX is fixed to the circle which rotates uniformly (with constant speed) in an anticlockwise direction about O.

At the same time the line OP rotates in a clockwise



direction at the same rate as the circle. Suppose Q is the point where OP' and A'X' intersect, where OP', A'X' are the positions taken up by OP, AX at a certain time instant. Express the coordinates of Q in terms of the angle of rotation and hence find the equation of Q. Sketch the locus.

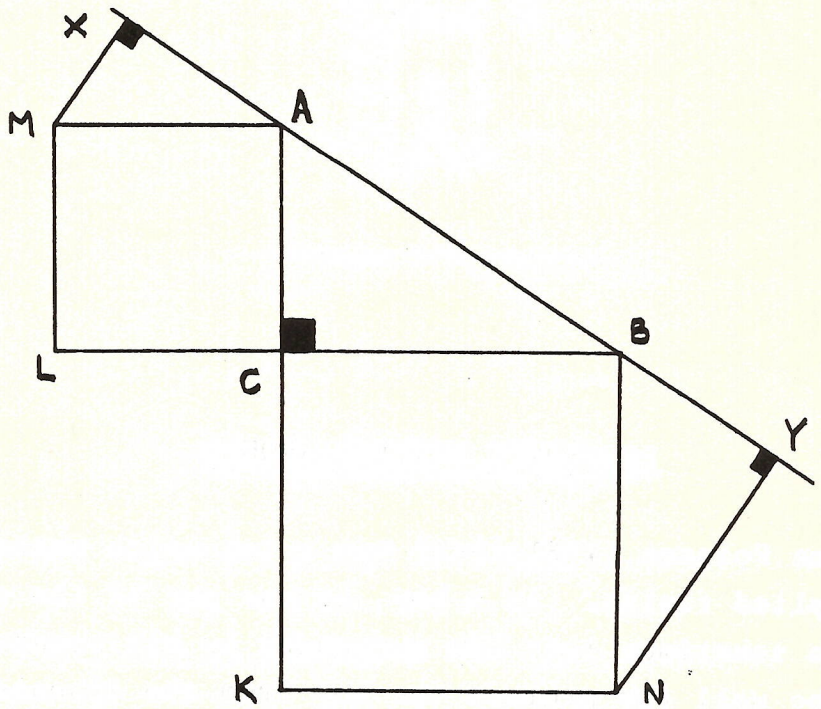
047. Prove that

$$\binom{2}{2}\binom{4}{2} \cdots \binom{2n-2}{2}\binom{2n}{2} = n!\ 1.3.5....(2n-1)$$

$$= \frac{1}{2^n}(2n)!$$

Interpret the equalities by a combinatorial meaning. [Hint: consider pairing off 2n players in a tennis tournament.]

048.



The triangle ABC has a right angle at C. AMLC and CBNK are squares drawn on the sides AC and CB. MX and NY are perpendicular to the hypotenuse produced.

Prove that the sum of the lengths MX and NY is equal

to the length of the hypotenuse AB.

049. Let $k_1,k_2,\ldots,k_n$ be n integers and let $m_1,m_2,\ldots,m_n$ be a rearrangement of these integers. Show that the expression

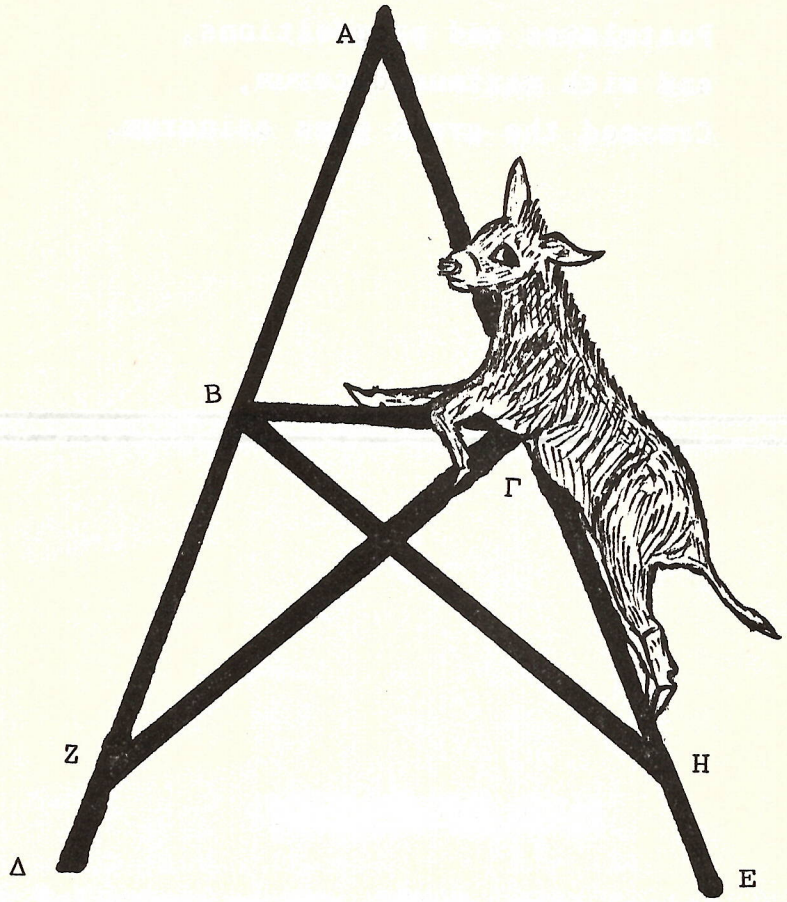
$$|k_1 - m_1| + |k_2 - m_2| + \ldots + |k_n - m_n|$$

is an even number.

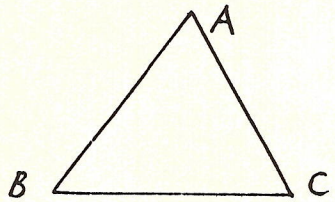
POISSON'S HANGING AROUND

Simeon Poisson (1781-1840) was a top-notch mathematician in the applied fields. As a boy, he was put in the care of a nurse, who sometimes left him suspended by his straps to a nail in the wall (to protect the boy from the disease and dirt of the floor!). Poisson said that his gymnastic efforts when thus suspended caused him to swing back and forth, and it was in this way that he became familiar early with the pendulum, the study of which occupied much of his later life.

THE BRIDGE OF THE ASSES



The diagram (with the Greek letters, but without the ass) accompanies Proposition 5 in Book 1 of the celebrated "ELEMENTS" of Euclid. It is used for proving that the angles on the base of the isosceles triangle are equal. We may mention here that a more elegant proof was given later by Pappus (c.300 A.D.). Incidentally, this is also the proof supplied by a computer programmed to do geometric proofs.



Given that AB = AC , we compare triangles BAC and CAB . These are congruent (S.A.S.), hence $m\angle ABC = m\angle ACB$.

However, since the training in geometry, until recent times, was based on a strict adherence to the books of Euclid in most countries, particularly in England, the proof of proposition 5 was regarded as the first serious hurdle on the road to formal geometry, a hurdle in which "asses" were left behind. Hence the Latin name: "PONS ASINORUM", the bridge of the asses.

I know all the definitions,
Postulates and propositions,
and with maximum decorum,
Crossed the great pons asinorum.

FALSE CANCELLATION

Have you noticed that 16/64 can be cancelled falsely to give the correct answer.

$$\frac{1\not{6}}{\not{6}4} = \frac{1}{4} .$$

If we restrict our attention to two-digit numbers then there are only four more such possibilities:

$$\frac{1\not{9}}{\not{9}5} = \frac{1}{5} , \quad \frac{2\not{6}}{\not{6}5} = \frac{2}{5} , \quad \frac{4\not{9}}{\not{9}8} = \frac{4}{8}$$

and the trivial cases of

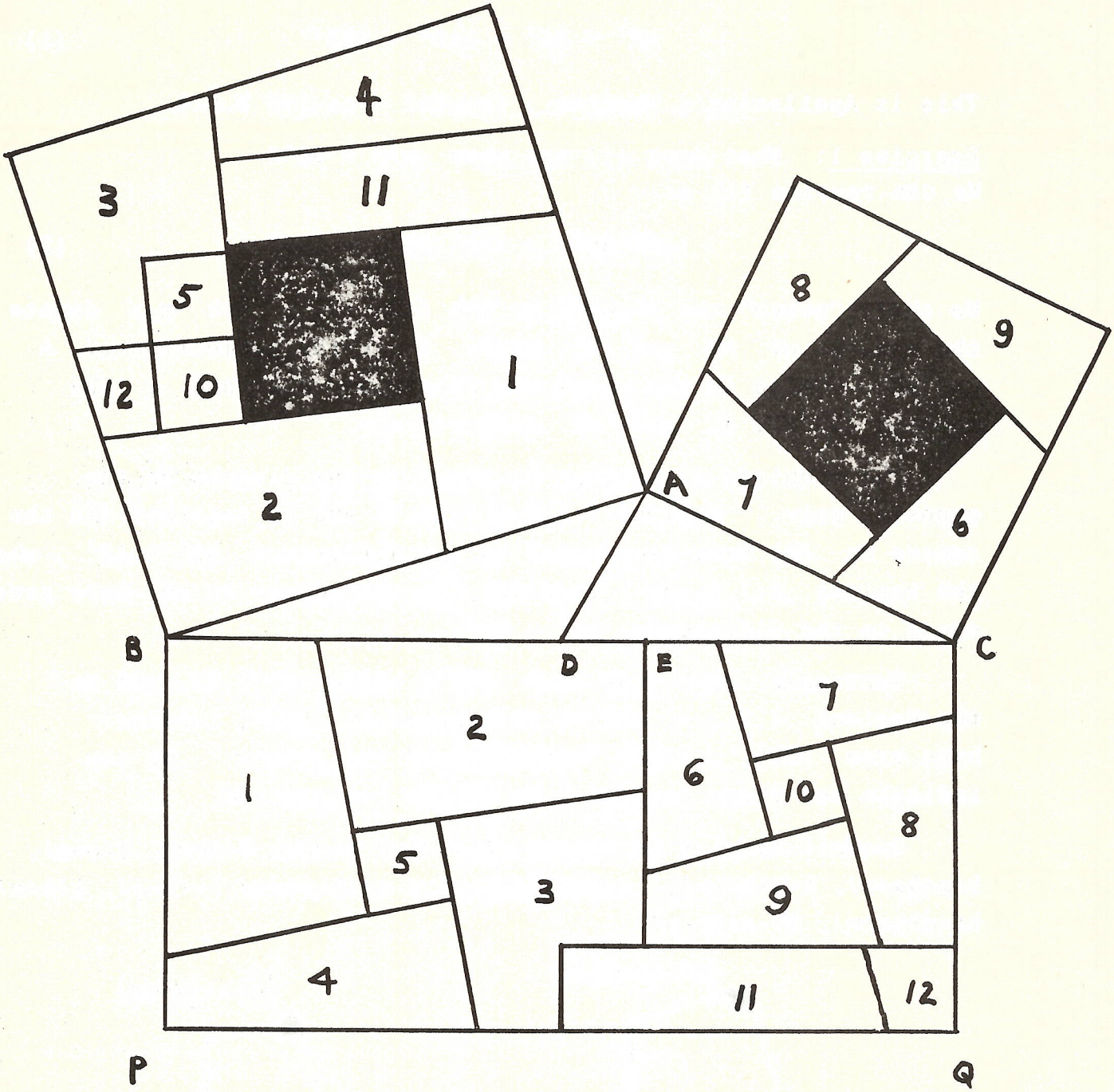
$$\frac{11}{11} , \quad \frac{22}{22} , \quad \frac{33}{33} , \quad \frac{44}{44} , \text{ etc.}$$

What happens if we take three-digit numbers? There are the examples:

$$\frac{1\not{9}\not{9}}{\not{9}\not{9}5} = \frac{1}{5} , \quad \frac{1\not{6}\not{6}}{\not{6}\not{6}5} = \frac{1}{5} .$$

Can you find any more? Can you find all such pairs of three-digit numbers? What about four-digit numbers? What about n-digit numbers?

Theorem of Apollonius



THE THEOREM OF APPOLONIUS BY DISSECTION (OR A PROJECT FOR THE MATHEMATICS CLUB)

Let D be the midpoint of the base BC of a triangle ABC , then

$$AB^2 + AC^2 = 2AD^2 + 2BD^2 \qquad (1)$$

This is Apollonius's Theorem. (ca.262 - ca.190 B.C.)

Exercise 1: What does (1) say when m/A = 90°?
We can rewrite (1) as:

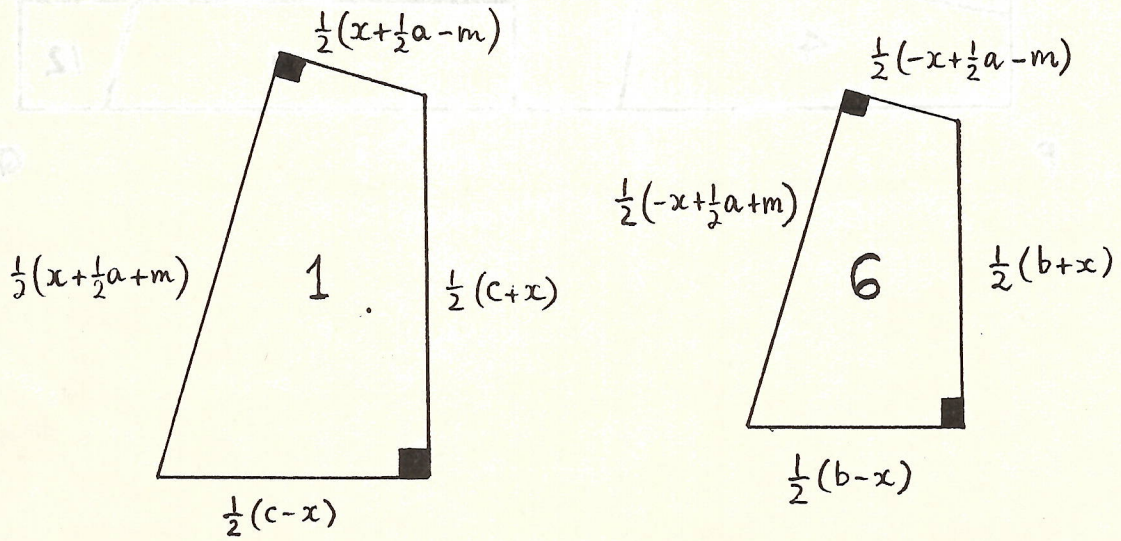
$$(AB^2-AD^2) + (AC^2-AD^2) = 2BD^2 \qquad (2)$$

We shall "prove" the theorem in the form (2). Let m,x denote the lengths of AD and DE where E is the projection of A on BC . Also in the usual way denote by a,b and c the lengths of BC, AC and AB respectively.

The required dissection is shown in the diagram on the opposite page. Regions 1,2,3,4,5,10,11 and 12 together (in the square on AB) have area equal to the difference $AB^2 - AD^2$, whereas the regions 6,7,8 and 9 together (in the square on AC) have area equal to the difference $AC^2 - AD^2$.

Rectangle BCQP has area $2BD^2$, since BP = BD, and so the Regions 1,2,..., and 12 rearranged in BCQP, together have area $2BD^2$. This "establishes" the result. We now indicate how these regions 1 to 12 are constructed.

Quadrilateral 1 is constructed with sides $\frac{1}{2}(c+x)$, $\frac{1}{2}(c-x)$, $\frac{1}{2}(x+\frac{1}{2}a+m)$, $\frac{1}{2}(x+\frac{1}{2}a-m)$; *quadrilateral 6 with sides* $\frac{1}{2}(b-x)$, $\frac{1}{2}(b+x)$, $\frac{1}{2}(-x+\frac{1}{2}a+m)$, $\frac{1}{2}(-x+\frac{1}{2}a-m)$.



The square on AB is then dissected into four quadrilaterals equal to 1, plus the shaded square (AD^2); similarly for the square on AC. These quadrilaterals are reassembled in the rectangle BCQP (BQ = BD = $\frac{1}{2}a$) in the square on BE, EC respectively with the right-angled corners in the reverse positions, so as to enclose squares each equal to DE^2 (5 and 10 are $x \times x = DE \times DE$). Finally two quadrilaterals are subdivided (4,11; 11 is x units wide) and (3,5,10,12) to fit in as shown in the figure.

To avoid difficulties of overlapping in this method of dissection the triangle should be drawn with AD smaller than either AB or AC.

Exercise 2: Trace the diagram shown, cut up the squares on AB and AC and fit the non-shaded pieces as shown into the rectangle BCQP.

Exercise 3: Construct a model of the diagram shown using lino tiles or three-ply. However, select your own values of a,b,c,m,x.

Exercise 4: Justify algebraically that the above dissection works.

Exercise 5: Find an algebraic proof (using Pythagoras's Theorem) of Appolonius's theorem.

[Adapted from: G.D.C. Stokes, Edinburgh Mathematical Notes 24, xviii (1929)]

A MIXED UP MATHEMATICIAN

Newton verified and added to the inverse square law of gravitation (the force between two particles is inversely proportional to the square of the distance between them). Augustus De Morgan summed up the situation by saying:

The notion of gravitation was not new;
but Newton went on.

THE FOUR FOURS PROBLEM

Here is a challenging and enjoyable pastime dating back to 1881. We wish to form as many whole numbers as possible by using only the digit 4 four times - no more, no less - together with simple mathematical symbols. Addition, subtraction, multiplication and division are of course admissible, and immediately give us:

$$1 = \frac{4}{4} + 4 - 4$$

$$2 = \frac{4}{4} + \frac{4}{4}$$

$$3 = \frac{4 + 4 + 4}{4} .$$

The use of brackets and 44 further yield,

$$4 = 4(4 - 4) + 4$$

$$5 = \frac{(4\times4) + 4}{4}$$

$$6 = 4 + \frac{4 + 4}{4}$$

$$7 = \frac{44}{4} - 4 , \text{ etc.}$$

The square root sign admits

$$11 = 44/(\sqrt{4} + \sqrt{4})$$

$$12 = (44 + 4)/4$$

$$13 = \frac{44}{4} + \sqrt{4} , \text{ etc.}$$

These expressions are not necessarily unique. Some numbers can be expressed in several different ways. In order to get 19 we need additional symbols; either the factorial sign $n! = 1 \times 2 \times 3 \ldots \times n$, in which case, $19 = 4! - 4 - \frac{4}{4}$, or the decimal point $\cdot 4 = \frac{4}{10}$ and the repeating decimal $\dot{\cdot}4 = \cdot4444\ldots = \frac{4}{9}$, in which case,

$$19 = \frac{4}{.4} + \frac{4}{\dot{\cdot}4}$$

These last three symbols lead to a great many more formations,

$$24 = 4! - 4 + \sqrt{4} + \sqrt{4}$$

$$25 = 4 \times 4 + \frac{4}{\dot{\cdot}4}$$

$$26 = 4 \times 4 + \frac{4}{.4} , \text{ etc.}$$

The only other operation we shall allow is that of raising to a power, i.e. 4^4, $4^{4!}$, etc.

So, $48 = 4^{4!} - 4 \times 4$

$264 = 4^4 + 4 + 4$, etc.

Using the symbols and operations described some numbers can, but others cannot be formed. There seems to be no way to express 113 within these restrictions unless we use a bizarre combination:

$$113 = \frac{4!}{\sqrt{4}} + \frac{\sqrt{4}}{\cdot 4},$$

where $\dot{\sqrt{4}}$ is meant to be $\dot{2} = \cdot 2222\ldots = \frac{2}{9}$.

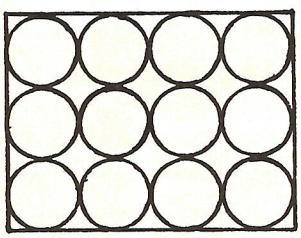
REFERENCE: Scientific American, January 1964.

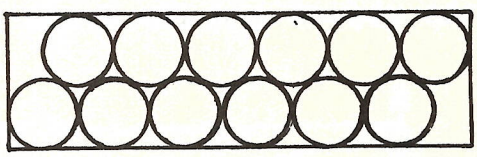
Research Corner

THE TWELVE PENNIES PROBLEM

Here is an unsolved problem which you may like to try. Twelve pennies (20 cent pieces will do, for those with short memories!) can be packed without overlapping in a 3 × 4 lattice point array as shown below. If each coin has diameter 1, its area is $\frac{\pi}{4}$. We can think of each coin lying in a square of unit area; it follows that the twelve pennies cover $\frac{\pi}{4}$ of the area of the drawn rectangle.

It is conjectured that the twelve coins cannot be packed into a smaller rectangle. For example, the 2 × 6 lattice





arrangement is no better and no worse. A candidate for 'next best' seems to be the 2 × 6 skew-lattice arrangement, shown above right; but its area is slightly greater than 12.

The problem then is either to produce a smaller rectangle which contains the pennies or to prove that no such rectangle has area less than 12.

DEPTH OF FIELD

AN APPLICATION OF GEOMETRY TO PHOTOGRAPHY

Throughout this article we shall consider a camera with a simple, thin convex lens as shown in figure 1. We shall assume some physics:-

(i) Light entering a lens, parallel to the axis of the lens, passes through a point F (the focus) at a fixed distance f (the focal length) behind the lens.

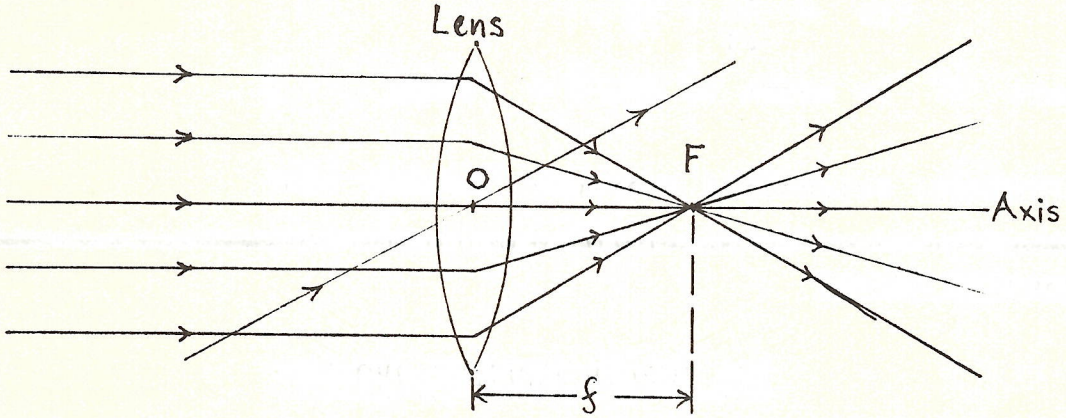


fig. 1.

(ii) A light beam entering the lens and passing through O, reemerges undeflected.

(iii) All light rays emitted from a point, on passing through the lens focus come together at the same point behind the lens.

§1. The Lens Equation

First using the properties of similar triangles we establish the fundamental equation connecting the distance of the object and the image from the lens.

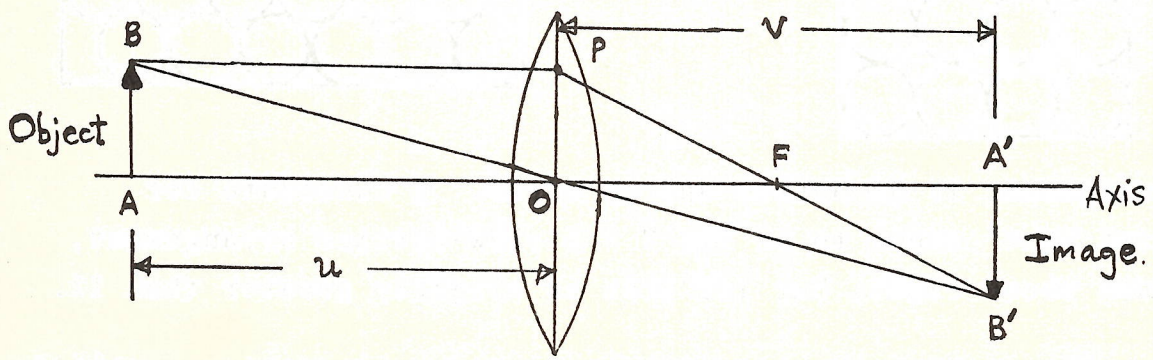


fig. 2.

The light ray $\overrightarrow{BP}$ is parallel to the axis and so by (i) is deflected through the focus F . $\overrightarrow{BO}$ passes through the centre of the lens and so by (ii) is undeflected. Let B' be the point where $\overrightarrow{BO}$ produced and $\overrightarrow{PF}$ produced meet. B' is the image of B . Let AO = u and OA' = v.

Now triangles ABO and A'B'O are <u>similar</u> and so:

$$\frac{B'A'}{BA} = \frac{A'O}{AO} = \frac{v}{u}$$

Also triangles OPF and A'B'F' are <u>similar</u>; hence

$$\frac{B'A'}{BA} = \frac{B'A'}{OP} = \frac{A'F}{OF} = \frac{v-f}{f}$$

Hence

$$\frac{v}{u} = \frac{v-f}{f} = \frac{v}{f} - 1$$

Dividing throughout by v

$$\frac{1}{u} = \frac{1}{f} - \frac{1}{v}$$

or

$$\boxed{\frac{1}{u} + \frac{1}{v} = \frac{1}{f}} \qquad \ldots(1)$$

§2. Diameter of the Circle of Confusion

Let us place a negative (or a screen) behind the lens.

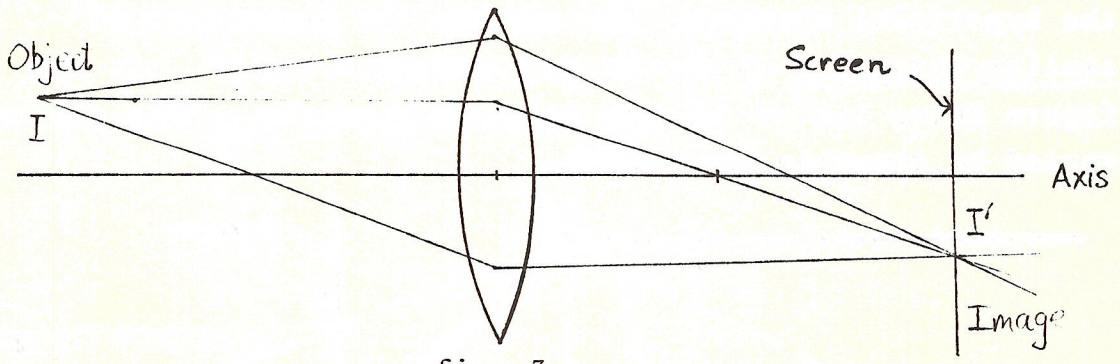


fig. 3.

If the image on the screen is a point whenever the source is a point source, then we say that the image is in focus on the screen. If not we get a "circular" patch of light on the screen as in figure 4.

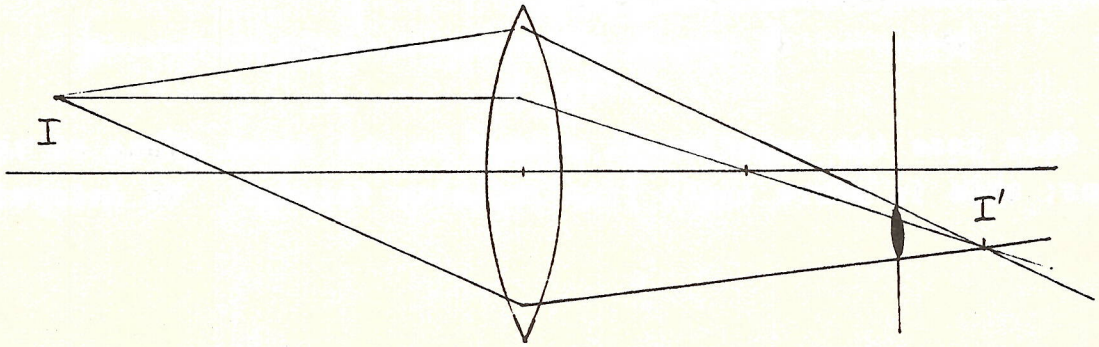


fig. 4.

We say that in the situation of figure 4 that I' is out of focus. If the object is an extended body (e.g. a brick) then we cannot expect that every point of the image will be in focus on the screen! (In figure 3, a point further out from the lens that I will be in focus closer to the lens that I' and viceversa for a point closer to the lens.)

However we can tolerate some out-of-focusness because of the limitations of the human eye. Let c be the diameter of the largest out-of-focus patch of light on the negative such that when the negative is developed and an enlarged print is made which is then viewed from about 25 cm, all points on the print appear sharp. The number c is referred to as the diameter of the circle of confusion. For a 35 mm negative which is subsequently made into a 6×8 in (21 cm × 28 cm) print, an acceptable value for c is 0.03 mm.

§3. The Hyperfocal Distance, h

Once the number c has been determined, the following question can be asked. What is the shortest distance from the lens that an object can be placed so that the object and all points beyond it are "in focus"? This distance is called the hyperfocal distance. We can use the properties of similar triangles to obtain an expression for it. In the figure below let d be the aperture diameter. This can be varied by changing the f-stop.

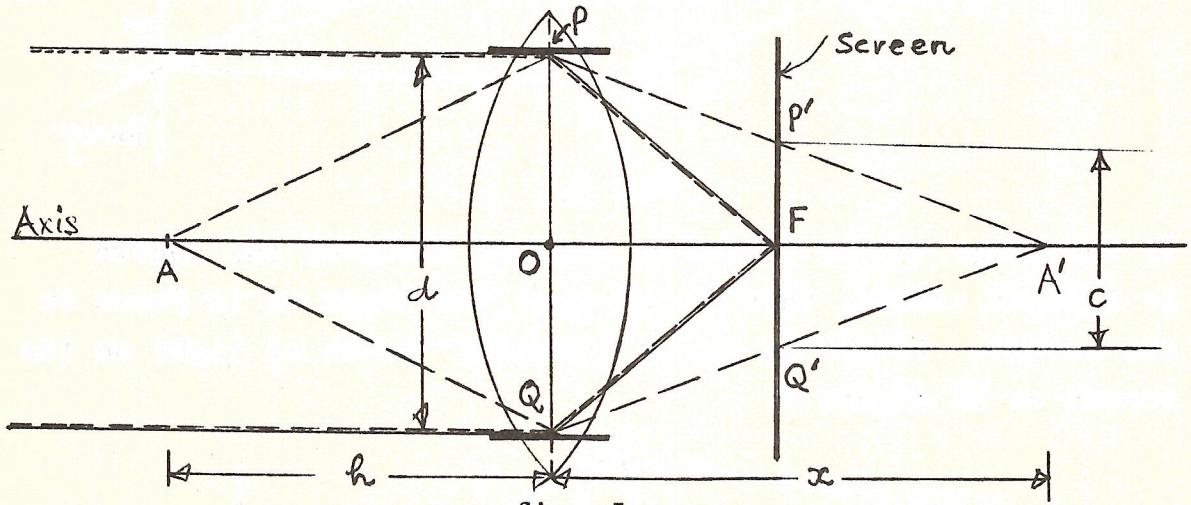


fig. 5.

In this case the negative is placed on the focal plane of the lens; this is where light from infinity focuses. We determine

h such that $P'Q'$ is the allowable c. By the Lens Equation:

$$\frac{1}{h} + \frac{1}{x} = \frac{1}{f} \qquad \ldots(2)$$

and since triangles $A'P'F$ and $A'PO$ are similar

$$\frac{P'Q'}{PQ} = \frac{P'F}{PO} = \frac{A'F}{A'O},$$

that is:

$$\frac{c}{d} = \frac{x-f}{x} \qquad \ldots(3)$$

Solving (3) for x we obtain:

$$x = \frac{df}{d-c} \qquad \ldots(4)$$

By equation (2):

$$\frac{1}{h} = \frac{1}{f} - \frac{1}{x}$$

and substituting in the value for x in (4):

$$\begin{aligned} \frac{1}{h} &= \frac{1}{f} - \frac{d-c}{df} \\ &= \frac{1}{f}\left[1 - \frac{d-c}{d}\right] \\ &= \frac{c}{fd} \end{aligned}$$

Hence $$h = \frac{fd}{c} \qquad \ldots(5)$$

When using a camera one can vary the value of d by varying the f-stop on the lens; so d is usually measured in terms of the f number. In fact

$$N = f/d$$

where N is the f-stop number (e.g. f/8), f is the focal length and d is the diameter of the lens. Substituting into (5) we get

$$\boxed{h = \frac{f^2}{Nc}} \qquad \ldots(6)$$

Table of values of $h\ (= \frac{f^2}{Nc})$ for $c = \cdot 03$ mm and varying values of f and N (in metres to 4 sig. figs.).

	f \ N	1·4	2	2·8	4	5·6	8	11	16
fisheye	6	·8571	·6000	·4286	·3000	·2143	·1500	·1091	·0750
wide-angle	28	18·61	13·07	9·333	6·533	4·667	3·267	2·376	1·633
	35	29·17	20·42	14·58	10·21	7·292	5·104	3·712	2·552
normal	50	59·52	41·67	29·76	20·83	14·88	10·42	7·576	5·208
telephoto	105	262·5	183·8	131·3	91·88	65·63	45·94	33·41	22·97
	135	433·9	303·8	217·0	151·9	108·5	75·94	55·23	37·97
	200	952·4	666·7	476·2	333·3	238·1	166·7	121·2	83·33

We can now already do some interpretation: if we choose a large f-stop number (e.g. f/32), then h is small and so we can expect a large depth of field. (See §4.)

§4. Depth of Field

When a camera is focussed on an object, there will exist a plane of sharpest focus (usually passing through the most important part of the object). However, there will also be a zone extending in front and behind this plane within which unsharpness of the image will be negligible and everything can be accepted as being in focus. The length of this zone is called the depth of field. Consider the following diagram:

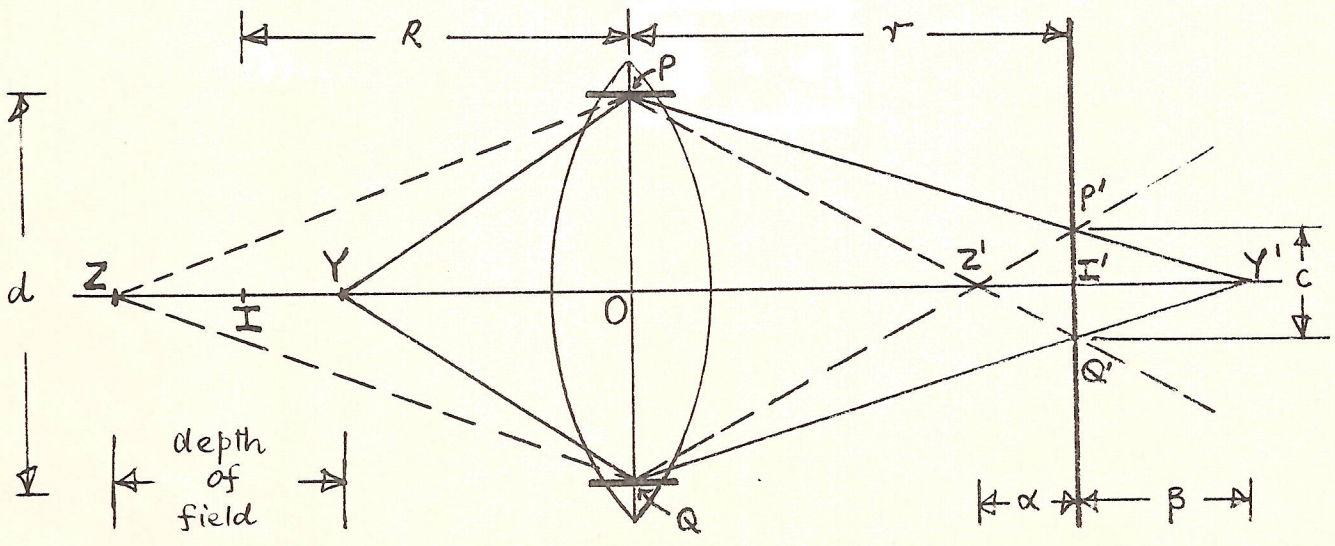


fig. 6.

The points Z,Y,I on the object focus at Z',Y',I' and the negative is placed at I' (assumed to be the point at which I is best focussed).

By the lens equation:

$$\frac{1}{R} + \frac{1}{r} = \frac{1}{f} \qquad \ldots(7)$$

Letting R_1, R_2 be the distances of Z,Y respectively from the lens we also have by the Lens Equation

$$\frac{1}{R_1} + \frac{1}{r-\alpha} = \frac{1}{f} \qquad \ldots(8)$$

$$\frac{1}{R_2} + \frac{1}{r+\beta} = \frac{1}{f} \qquad \ldots(9)$$

where $\alpha = Z'I'$ and $\beta = I'Y'$.

Back to our <u>similar triangles</u>. Since triangles POY' and P'I'Y' are <u>similar</u>

$$\frac{I'Y'}{OY'} = \frac{P'I'}{PO} = \frac{P'Q'}{PQ}$$

that is:

$$\frac{\beta}{r+\beta} = \frac{c}{d} \qquad \ldots(10)$$

Also triangles Z'P'Q' and Z'PQ are similar; hence:

$$\frac{Z'I'}{OZ'} = \frac{P'Q'}{PQ}$$

that is:

$$\frac{\alpha}{r-\alpha} = \frac{c}{d} \qquad \ldots(11)$$

From equations (7), (8), (9), (10) and (11) we now wish to calculate $R_1 - R_2$ which is the depth of field. The answer is in terms of R,f and h is:

$$R_1 - R_2 = \frac{2hR(R-f)}{h^2-(R-f)^2} \qquad \ldots(12)$$

where we have used equation (5), viz., $hc = fd$.

[It will be a good exercise in algebraic manipulation to verify this last expression. Here is how you go about it:

By equation (10) and (11)

$$\beta = \frac{rc}{d-c} \quad \textit{and} \quad \alpha = \frac{rc}{d+c} .$$

whence

$$r + \beta = \frac{rd}{d-c} \quad \textit{and} \quad r - \alpha = \frac{rd}{d+c}$$

$$\therefore \quad \frac{1}{r-\alpha} = \frac{d+c}{d} \cdot \frac{1}{r} = \frac{d+c}{d}\left(\frac{1}{f} - \frac{1}{R}\right) \quad \textit{by equation (7).}$$

Similarly

$$\frac{1}{r+\beta} = \frac{d-c}{d}\left(\frac{1}{f} - \frac{1}{R}\right)$$

By equations (8) and (9), and using $fd = hc$

$$\frac{1}{R_1} = \frac{1}{f} - \frac{1}{r-\alpha} = \frac{1}{f} - \frac{d+c}{d}\left(\frac{1}{f} - \frac{1}{R}\right) = \frac{fc+fd-Rc}{fdR}$$

$$= \frac{fc+hc-Rc}{hcR} = \frac{f+h-R}{hR}$$

$$\frac{1}{R_2} = \frac{1}{f} - \frac{1}{r+\beta} = \frac{h-f+R}{hR}$$

$$\therefore \quad R_1 - R_2 = hR\left[\frac{1}{h+f-R} - \frac{1}{h-f+R}\right]$$

$$= \frac{2hR(R-f)}{(h+f-R)(h-f+R)} = \frac{2hR(R-f)}{h^2-(R-f)^2} \; . \;]$$

We can simplify equation (12) by replacing $R - f$ by R as f is usually small compared with R. [However, for close-up work f does become significant.]

Then:

$$R_1 - R_2 \cong \frac{2hR^2}{h^2-R^2} = \frac{2f^2R^2}{Nc(h^2-R^2)} \qquad \ldots(13)$$

Of course (13) is valid only for $R < h$. For $R \geqslant h$ we know that the depth of field is infinite by the definition of h.

We conclude this article with some exercises for your investigation.

Exercise 1: Find out more about how c is related to the limitations of the human eye, on how many times you enlarge the negative to obtain a print, and on the way the final result is viewed. Ask you teacher to help you.

Exercise 2: Consider figure 6. If we require that ZY is the depth of field, what value of R should be chosen so that this will be the depth of field? In other-words what should be the distance setting on your camera so that Z and Y are just within the depth of field? (Obtain a formula expressing R in terms of R_1 and R_2.)

Exercise 3: List the various ways in which you can increase or decrease the depth of field.

Exercise 4: For $f = 50$ mm and $c = \cdot 03$ mm plot the graphs of $R_1 - R_2$ against R for $N = 2, 8, 11$.

SOLUTIONS TO PROBLEMS IN TRIGON VOL. 14, NO. 3

SENIOR PROBLEMS

036. Good solution by Mark Lawrence (Year 11, Westminster School). The above solution displays the socalled Pigeon Hole principle, namely, that if you try to pack $n + 1$ pigeons in n cages then one cage must contain 2 pigeons. Mark observed that if all ten teams could have played different number of matches, then in some order the teams must have played

$$0, 1, 2, 3, 4, 5, 6, 7, 8, 9$$

matches respectively. This leads to a contradiction in two ways: (i) between the team that played 0 and the team that played 9 matches; (ii) using part (b).

037. Good solution from M. Lawrence.

038. Good solution from M. Lawrence. Mark used an argument to show that if any equilateral triangle is given, then no angle θ can be found such that if the equilateral triangle is rotated through θ about its centroid placed at the centre of ellipse, that the three vertices of the triangle will fall on the ellipse.

039. Good solution from M. Lawrence.

040. Partial solutions from M. Lawrence and D.L. Garrard, Matriculation (Clare H.S.).

041. Partial solution from M. Lawrence.

042. Partial solution from M. Lawrence.

JUNIOR PROBLEMS

37. Good solution from Maxine Scelwyn (Woodlands C.E.G.G.S.).

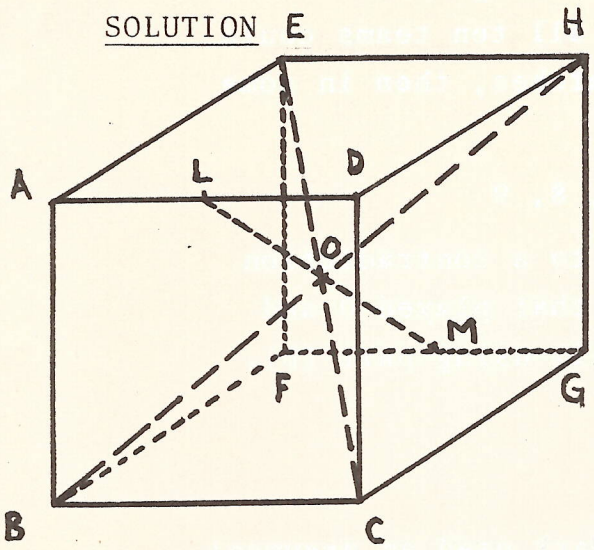
41. Correct solution from Maxine Scelwyn.

SOLUTIONS TO PROBLEMS IN VOL.14 NO.3

JUNIOR SECTION

36. Consider a line 1 joining the two midpoints of a pair of opposite edges of a cube. A cube has four diagonals; show that 1 is perpendicular to two of them.

SOLUTION



Let L,M be the mid points of the opposite sides AD and FG respectively. Clearly the midpoint of LM is the centre of the cube O , which is also the intersection point of the diagonals.

To prove that the diagonal BH is perpendicular to LM, we will show that the quadrilateral LHMB is a rhombus. LH = HM = BM = BL since these sides belong to the congruent triangles LHD, HMG, MBF and BAL respectively. Since the diagonals of a rhombus are perpendicular to each other, LM is perpendicular to BH . Similarly we can show that LM is perpendicular to EC . (It can be seen easily that LM is not perpendicular to AG and DF .)

37. Three golfers, named Tom, Dick and Harry are walking towards their clubhouse. Tom, the best golfer of the three always tells the truth. Dick sometimes tells the truth, while Harry, the worst golfer, always lies. One of the golfers wears a red pullover, the other a green one and the third a blue pullover. The golfer wearing the red pullover says that the golfer in the green pullover is Harry. The golfer in the green pullover introduces himself as Dick, while the golfer wearing the blue pullover states that the wearer of the green pullover is Tom. Who is who?

SOLUTION

Name for the moment the three golfers R,G and B . (R for red, G for green, B for blue). Their statements are listed in the table:

R	G	B
G = Harry	G = Dick	G = Tom

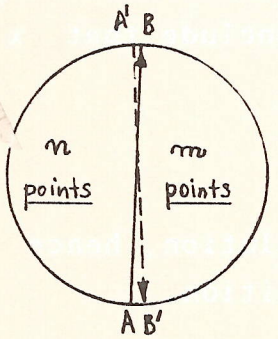
Since the three statements are contradictory, only one of the three is correct and this statement must be made by Tom.

Thus G cannot be Tom, for then the statement by B is true, and so Tom would have to be B also! Also G cannot be Dick because then G who is Dick, tells the truth contradicting again the conclusion that Tom is the only truth-teller in this case.
Hence G must be Harry. Then R is Tom, and B is Dick.
We may check that Tom tells the truth and Dick and Harry both lie.

38. The interior of a circle contains 2 million points, no two lying on any diameter. Must there exist a diameter which contains on each side of it exactly one million points?

SOLUTION

The answer is yes. We begin by drawing a diameter AB which does not contain any of the points. This is possible since there are an infinite number of diameters, but only 2,000,000 of them contain one point each.



Looking in the direction $\vec{AB}$ there are m points to the right and n points to the left. We may assume that $m \neq n$, for if they are equal, there is nothing to prove.

Now $m + n = 2{,}000{,}000$

so m and n are both even or both odd, hence the difference $m - n$ is certainly even. Rotate the diameter AB by 180°, so that it reaches the position A'B' in which there are n points to the right and m points to the left, i.e., the difference between the number of points to the left and right changed sign.

Now, whenever the rotating diameter sweeps past one point, either the number of points on the left increases by 1 and the number of points to the right decreases by 1, or the opposite happens. In any case the difference between the number of points on the

left and right changes by exactly 2. Thus during the rotation the difference must attain every even number between m - n and n - m at least once. Hence the diameter will at least once sweep into a region where the difference is 0, i.e., the number of points on the left is the same as the number on the right.

39. Find all positive integers m less than 1000 such that 2m is the original m with its digits in the reverse order.

SOLUTION

Discuss two cases:

(a) m has two digits:

$$m = 10x + y$$
$$2m = 10y + x$$

Subtracting:

$$m = 9(y-x)$$
$$\text{i.e.} \quad 9(y-x) = 10x + y \qquad (*)$$

It follows that m is a multiple of 9. But a number is divisible by 9 if and only if the sum of its digits is divisable by 9 . Thus, $x + y$ is a multiple of 9 . But as $x \leqslant 9$ and $y \leqslant 9$, $x+y = 9$ or 18 . If $x+y = 18$ then $x = 9$ and $y = 9$ which would imply that $m = 0$ which is not possible. We thus conclude that $x + y = 9$. Thus (*) becomes

$$9(9-x-x) = 10x+9-x$$

Simplified: $27x = 72$. This has no integer solution, hence there is no two digit number satisfying the condition.

(b) m has three digits:

$$m = 100x + 10y + z$$
$$2m = 100z + 10y + x$$

Then:

$$m = 99(z-x)$$
$$\text{i.e.,} \quad 100x + 10y + z = 99(z-x) \qquad (**)$$

Thus m is a multiple of 99.

In this case $y = 9$ by examining all multiples of 99 less than 1,000. Since $x+y+z$ must be divisible by 9 (for the same reasons as above) and since $y = 9$, $x+z$ is also divisible by 9.

Again we must have $x+z = 9$ else $m = 0$.
Simplified: $3x = 8$. Again there is no integer solution.

[Question: What about 4 digit numbers and beyond?]

40. The integers a and b are coprime (that is, they have no common divisors other than 1 and -1). Prove that $a + b$ and ab are coprime. Is the converse true?

SOLUTION

Suppose that p is a prime which divides the product ab. Since a,b are coprime, p divides exactly one of the factors. Say that p divides a, then p does not divide b; hence p does not divide the sum $a + b$. So no prime factor of the product ab divides $a+b$. Hence ab and $a+b$ are coprime.

Suppose that we know that ab and $a+b$ are coprime. Let p be a divisor of a. Then p divides ab hence p does not divided $a+b$, known to be coprime to ab. Since p divides a, but not $a+b$, p cannot divide b. So no factor of a can divide b. Thus a,b are coprime and so the converse theorem is also true.

41. James spent 3 days in town. On the first day he spent $\frac{1}{2}$\$ more than half of his money, on the second day he spent $\frac{1}{3}$\$ more than $\frac{1}{3}$ of his remaining money and on the third day $\frac{1}{4}$\$ more than $\frac{1}{4}$ of the remainder. He then returned home with \$20 in his pocket. How much did he spend in town?

SOLUTION

Let x be the amount of money that James brought to town.

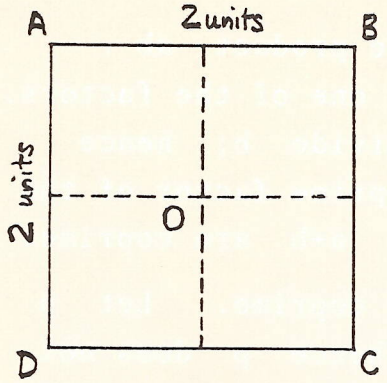
	Spending	Remainder
First day:	$\frac{x}{2} + \frac{1}{2}$	$x-(\frac{x}{2}+\frac{1}{2})=\frac{x-1}{2}$
Second day:	$\frac{1}{3}(\frac{x-1}{2})+\frac{1}{3}$	$\frac{x-1}{2}-(\frac{x-1}{6}+\frac{1}{3})=\frac{x-2}{3}$
Third day:	$\frac{1}{4}(\frac{x-2}{3})+\frac{1}{4}$	$\frac{x-2}{3}-(\frac{x-2}{12}+\frac{1}{4})=\frac{x-3}{4}$

Hence $\frac{x-3}{4} = 20$ or $x = 83$.

James came to town with $83 and spent $42, $14 and $7 on the first, second and third days respectively.

42. Suppose that there are 5 points inside a square with sides of length 2. Show that ther must be two of these points within $\sqrt{2}$ units of each other. Can you generalize this result?

SOLUTION



Divide the given square into 4 squares by drawing lines parallel to AB and BC through O (the midpoint of the square). Since we have 5 points inside the square ABCD, at least one of the squares contain 2 of the points, i.e., 2 points must be inside or on the circumference of one of the squares. The longest distance joining any two points within a square is the diagonal of the square. The diagonal of each of the small squares is $\sqrt{2}$ units. Hence the distance of two points within that square is $\leqslant \sqrt{2}$.

Generalisation: Suppose there are $n^2 + 1$ points inside (or on the circumference) of a square with sides of length n. Then there must be two of these points within $\sqrt{2}$ units of each other.

This can be seen by dividing the square into n^2 squares 1 unit × 1 unit and proceeding as above.

OPEN SECTION

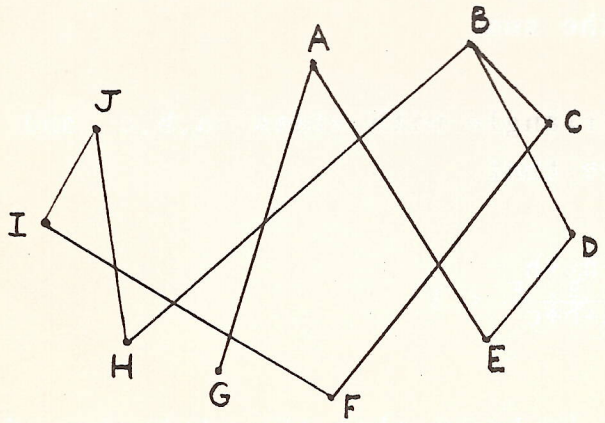
036. Ten teams have completed part of the football season, where every team is to play each team only once, though matches are not necessarily all played on the same day of the week. Show that

(1) At least two teams have played the same number of matches.

(2) The number of those teams which have played an odd number of matches is even.

SOLUTION

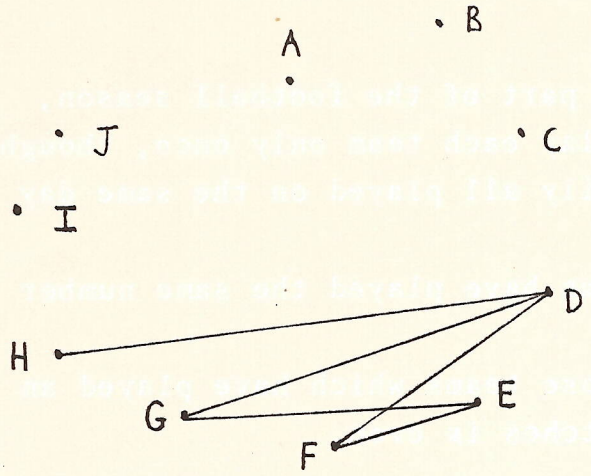
The situation at any time during the season may be represented by a diagram (graph) as shown below. The vertices A,B,C,D, E,F,G,H,I,J represent the teams. Two vertices are joined by a line (edge) if a match between the teams has been played to date.



For example, our graph shows that matches have been played by A-E,A-G,B-C,B-D, B-H,C-F,D-E, F-I, H-J and I-J.

(1) (a) Assume first that each team has played at least one match to date, i.e., that there is no isolated vertex in the graph. Since each team has played each other team at most once, no more than 9 edges could issue from each vertex. Since there are 10 vertices, from each of which it is possible to draw 1,2,..., or 9 edges, there must be at least two vertices from which an equal number of edges are drawn.

(b) Assume now that there are some isolated vertices. By considering only the non-isolated vertices we can apply the same reasoning as in (a): there are



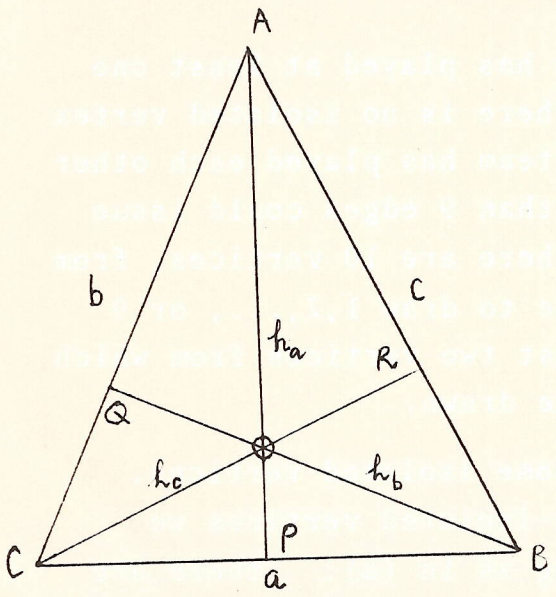
n(say) non-isolated vertices from each of which either 1,2,...,n-1 edges can be drawn. The result follows. [In the figure shown, A,B,C,I and J are isolated. From each of the remaining 5 non-isolated vertices up to 4 edges can be drawn.]

(2) Add up the number of matches played by each team (i.e., add up the number of edges issuing from each vertex). Then each match (edge) will be counted twice, hence the sum must be even. Thus it is impossible to have an odd number of odd terms in the sum.

037. ABC is an acute angled triangle with sides a,b,c and altitudes h_a, h_b, h_c prove that

$$\frac{1}{2} < \frac{h_a + h_b + h_c}{a+b+c} < 1$$

SOLUTION



It is known that the altitudes of a triangle concur at the point O (orthocentre) which is inside the triangle if the angles are all acute. Then

$$AO < AP = h_a$$
$$BO < BQ = h_b$$
$$CO < CR = h_c$$

Since the sum of the lengths of two sides of a triangle is greater than the length of the third, we have in triangle COB:

$$CO + BO > BC$$

Similarly:

$$AO + BO > AB$$
$$AO + CO > AC$$

By addition

$$2(AO + BO + CO) > BC + AB + AC = a + b + c$$

But

$$2(h_a + h_b + h_c) > 2(AO + BO + CO),$$

hence

$$2(h_a + h_b + h_c) > a + b + c$$

or

$$\frac{1}{2} < \frac{h_a + h_b + h_c}{a + b + c}$$

For the right hand side inequality consider the right angled triangles APC and APB . Since the hypotenuse must be the longest side we have

$$h_a < b \quad \text{and} \quad h_a < c$$

similarly,

$$h_b < a \quad \text{and} \quad h_b < c$$

and

$$h_c < a \quad \text{and} \quad h_c < b$$

By addition:

$$2(h_a + h_b + h_c) < 2(a + b + c)$$

$$\text{i.e.,} \quad \frac{h_a + h_b + h_c}{a + b + c} < 1$$

(Note that this part of the inequality is valid for <u>any</u> triangle.)

038. Show that no equilateral triangle inscribed in a non-circular ellipse can have its centroid coincide with the centre of the ellipse.

SOLUTION

Assume that the centre O of the ellipse of equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad a \neq b \qquad (1)$$

is also the centroid of the equilateral triangle PQR where P,Q and R are points on the ellipse. The centroid of the equilateral triangle is equidistant from the vertices, so

$$OP = OQ = OR = r .$$

This means that P,Q,R are amongst the intersections of the ellipse and the circle of which the equation is:

$$x^2 + y^2 = r^2 \qquad (2)$$

However, the intersections of an ellipse (non-circular) and a circle of common centre form a rectangle (if the points of intersection are real and distinct). No three vertices of a rectangle can form an equilateral triangle, hence our assumption leads to a contradiction.

NOTE: To show that the intersections form a rectangle, solve equations (1) and (2). We get

$$x^2 = \frac{a^2(r^2-b^2)}{a^2-b^2} = u \quad \text{say}$$

and

$$y^2 = \frac{b^2(a^2-r^2)}{a^2-b^2} = v \quad \text{say} .$$

If $b < r < a$, then $x = \pm\sqrt{u}$ and $y = \pm\sqrt{v}$ so we get $(\sqrt{u},\sqrt{v})$, $(-\sqrt{u},\sqrt{v})$, $(-\sqrt{u},-\sqrt{v})$ and $(\sqrt{u},-\sqrt{v})$ as the four solutions.

039. Each summer 10% of the trees in a certain plantation die out, and each winter, workmen plant 100 new trees. At the end of winter, there were 1200 trees in the plantation. How many living trees were there at the end of the winter of 1970? What happens to the number of trees in the plantation during the 21st century provided that the conditions remain unchanged?

SOLUTION

Let x_n be the number of trees in the plantation at the end of winter in year n. Then we have

$$x_{n+1} = .9\, x_n + 100 \qquad (1)$$

Let us use (1) to obtain x_{n+k} in terms of x_n,

$$\begin{aligned} x_{n+2} &= .9\, x_{n+1} + 100 = (.9)^2 x_n + 100(.9 + 1) \\ x_{n+3} &= .9\, x_{n+2} + 100 = (.9)^3 x_n + 100((.9)^2 + .9 + 1) \\ &\vdots \\ x_{n+k} &= \ \ldots \ = (.9)^k x_n + 100((.9)^{k-1} + \ldots + (.9)^2 + (.9) + 1) \end{aligned}$$

Formally this last line would need to be proved by mathematical induction.

Hence
$$x_{n+k} = (.9)^k x_n + 100\,\frac{(1-(.9)^k)}{1-.9}$$

$$= (.9)^k x_n + 1000\,(1-(.9)^k) \qquad (2)$$

$$= (.9)^k (x_n - 1000) + 1000 \qquad (3)$$

(Here we have used the summation formula for a geometric progression, namely that

$$a + ar + ar^2 + \ldots + ar^m = \frac{a(1-r^{m+1})}{1-r}$$

with $a = 100$, $r = .9$.)

Now $x_{1976} = 1200$. Solving (2) with $k = 6$ and $n = 1970$, we have

$$1200 = (.9)^6\,(x_{1970} - 1000) + 1000$$

whence $x_{1970} = \frac{200}{(.9)^6} + 1000 = 1376.3 = 1376$ trees.

when $n = 1976$ and $k = 24$

$$x_{2000} = (.9)^{24}\ 200 + 1000 = 1015 \text{ trees}$$

and when $k = 50$ and 51 we have

$$x_{2026} = 1001 \text{ trees}$$

and $$x_{2027} = 1000 \text{ trees.}$$

We now observe that once $x_n = 1000$ for some n then $x_m = 1000$ for all $m > n$ (by (3)). Hence during the 21st Century (from year 2027 on) the number of trees each spring stabilizes at 1000 trees.

040. Let f be a continuous function such that $f(x)$ is defined for $x \geqslant 0$ and

$$f[x+y)^2] = f(x^2]+f[y^2]$$
$$\text{and } f(1) = 1 \ .$$

Show that

(i) $f(0) = 0$

(ii) $f[(nx)^2] = nf[x^2]$

Determine the function f, giving your reasons.

SOLUTION

(i) Let $y = 0$, then

$$f[(x+0)^2] = f[x^2] + f[0^2]$$

by the given equation, hence

$$f[x^2] = f[x^2] + f[0]$$

It follows that $f(0) = 0$.

(ii) Use induction on n. When $n = 1$, then

$$f[(nx)^2] = f[(1x)^2] = f(x^2) = 1f(x^2) \ .$$

Assume that the relation holds for $n = m$. Then for $n = m+1$ we have

$$f[((m+1)x)^2] = f[(mx+x)^2] = f[(mx)^2]+f[x^2]$$

by our functional equation. Using the induction hypothesis:

$$f[(mx)^2] = mf[x^2]$$

we have

$$f[((m+1)x)^2] = m\ f[x^2] + f[x^2] = (m+1)f[x^2]$$

as required. By the principle of mathematical induction $f[(nx)^2] = n\ f[x^2]$ for all $n \in Z^+$. Apply this equation

now with $x = 1$. Then

$$f[(n.1)^2] = n\, f[1^2] = n\, f[1] = n$$

since $f[1] = 1$. That is

$$f(n^2) = n$$

Hence if $x = n^2$, $f(x) = f(n^2) = n = \sqrt{x}$. This suggests that the required function is

$$f(x) = \sqrt{x} \quad (x \geqslant 0)$$

We prove this now by establishing it first for the case when x is the square of a rational number

Let $x = \left(\frac{m}{n}\right)^2$.

Then $f(x) = f[\left(\frac{m}{n}\right)^2] = f[(m\left(\frac{1}{n}\right))^2] = mf[\left(\frac{1}{n}\right)^2]$ (*)

Now $1 = f(1) = f[\left(n.\frac{1}{n}\right)^2] = n\, f[\left(\frac{1}{n}\right)^2]$

Hence $f[\left(\frac{1}{n}\right)^2] = \frac{1}{n}$

From (*), $f(x) = m.\frac{1}{n} = \frac{m}{n} = \sqrt{x}$, as required.

Finally, let us accept that every positive number x can be approximated by squares of rational numbers. (Instead of proving this rigorously we point out that in practice we use this fact when finding a decimal approximation of the square root of a positive number. Thus for example 2 can be approximated by 1^2, 1.4^2, 1.41^2, 1.414^2,... and so on.) The relation

$$f(x) = \sqrt{x}$$

holds when x is the square of a rational number and $f(x)$ is known to be continuous. By the continuity argument discussed in the article on page 24 (Trigon, Vol.14,No.3), we conclude that the required function is

$$f(x) = \sqrt{x} \text{ for all } x \geqslant 0 .$$

041. It is known that $2^4 = 4^2$. Are there any other integers m,n such that

$$m^n = n^m \quad ?$$

SOLUTION

$m^n = n^m$ is certainly true if $m = n$.

This case is trivial and we may exclude it. Assume therefore that m,n are unequal and let m be the smaller of the two. Then

$$n = m + k$$

where $k \geqslant 1$ (k integer). Thus

$$m^{m+k} = (m+k)^m \ .$$

Dividing through by m^m we obtain

$$m^k = (1 + \frac{k}{m})^m \ .$$

Since the left hand side is an integer, so is the right hand side, so

$$\frac{k}{m} = r$$

where r is a positive integer. Now we have

$$m^{r\,m} = (1+r)^m$$

Hence $m^r = 1+r$ (*)

Here m must certainly be more than 1, so let

$$m = 1+s$$

where s is a positive integer. If $r > 1$, then

$$m^r = (1+s)^r = 1+rs+\binom{r}{2}s^2+\ldots = 1+r \ .$$

Since all the terms involved here are positive, this cannot hold. Thus $r = 1$.

Hence $m' = 1+1 = 2$ (using (*)), so $k = mr = 2$ and $n = m+k = 4$. This shows that the only non-trivial solution is the wellknown

$$2^4 = 4^2 \ .$$

042. Consider the family of curves:

$$|x|^n + |y|^n = 1$$

where n is a positive real number. Sketch the curves when $n=1$, $n=2$, $n=\frac{1}{2}$. What happens when $n \to 0$ and when $n \to \infty$? Show that for each $n > 0$ (n real) there exists a square the diagonals of which are the coordinate axes and the sides of which are tangential to the curve.

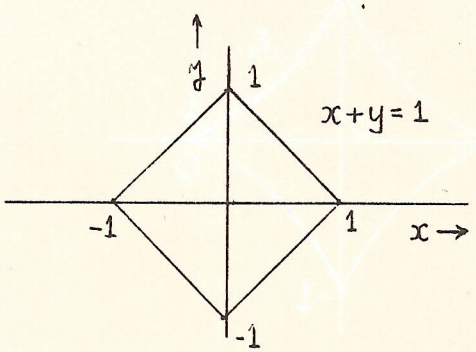
SOLUTION

Before sketching any of the required curves we observe that all of them have at least four axes of symmetry, namely, the two coordinate axes and the angular bisectors of the quadrants, since for any n, if (x_0, y_0) lies on the curve, so do $(-x_0, y_0)$, $(-x_0, -y_0)$ and $(x_0, -y_0)$.
Also $(\pm 1, 0)$ and $(0, \pm 1)$ are points on all of the curves.
Hence it suffices to know how the curve

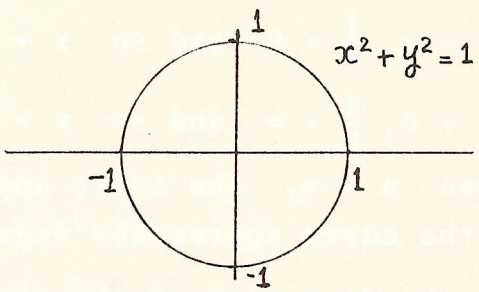
$$x^n + y^n = 1 \qquad (x \geqslant 0, y \geqslant 0)$$

behaves.

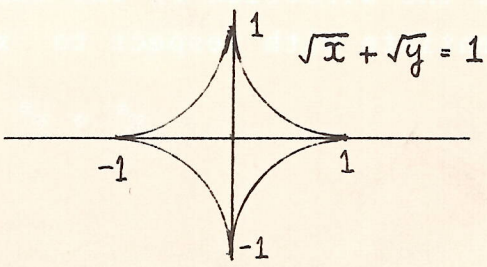
$n = 1$



$n = 2$



$k = \frac{1}{2}$



To find the shapes as $n \to 0$ and as $n \to \infty$, investigate the point where $x = y$ $(x > 0)$. Then

$$2x = 1 \quad \text{or} \quad x = (\tfrac{1}{2})^{\frac{1}{n}} \qquad \ldots (*)$$

For $n \geqslant 2$, the curve in the first quadrant lies inside the quadrilateral $SRQP_n$ in figure 1(a)

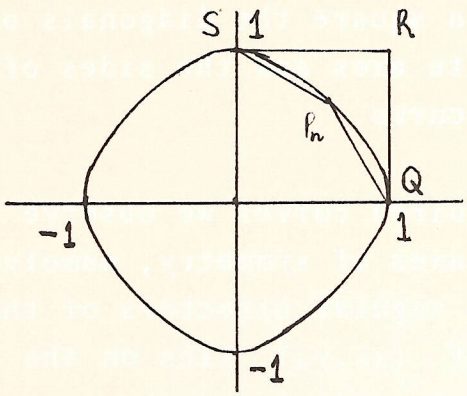


fig.1(a)

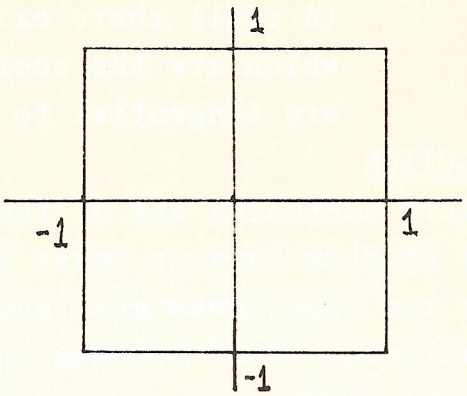


fig.1(b)

but for $n < 1$, the curve in the first quadrant lies inside the quadrilateral $SOQP_n$ in figure 2(a)

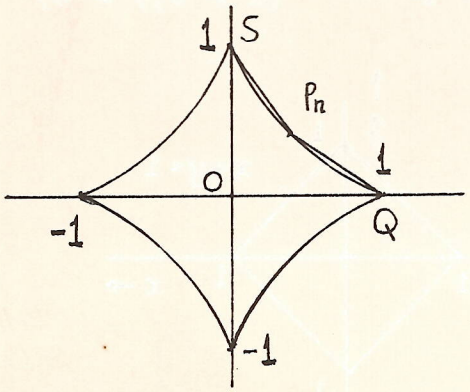


fig.2(a)

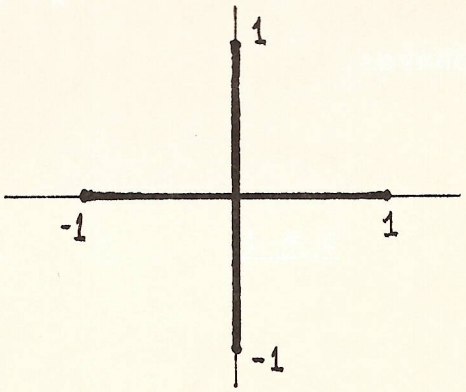


fig.2(b)

When $n \to \infty$, $\frac{1}{n} \to 0$ and so $x \to 1$ in equation (*)

when $n \to 0$, $\frac{1}{n} \to \infty$ and so $x \to 0$ in equation (*).

Thus when $n \to \infty$, the curve approaches figure 1(b) and when $n \to 0$, the curve approaches figure 2(b).

To find the direction of the tangent (in the first quadrant), differentiate with respect to x

$$x^n + y^n = 1$$

obtaining,

$$n\,x^{n-1} + n\,y^{n-1}\,\frac{dy}{dx} = 0$$

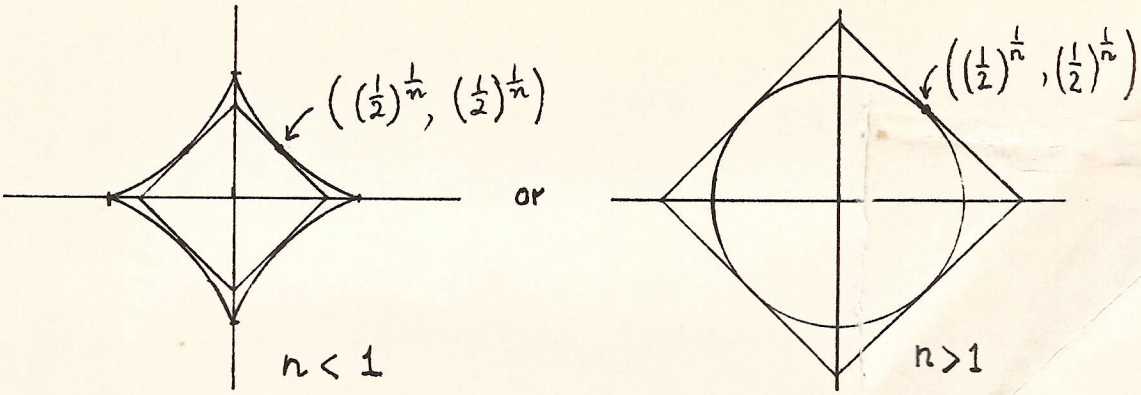
Hence

$$\frac{dy}{dx} = -\left(\frac{x}{y}\right)^{n-1}$$

When $x = y$, we have for all n

$$\frac{dy}{dx} = -1$$

This meams that for all the curves, the sides of the square passing through $(\pm(\frac{1}{2})^{\frac{1}{n}}, \pm(\frac{1}{2})^{\frac{1}{n}})$ and having the coordinate axes as diagonals will be tangential to the curve.



The curve is inscribed in the square when $n > 1$, but circumscribed to the square when $n < 1$ as shown above.